Maximal Solvable Subgroups

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Introduction

- Maximal solvable subgroup $M \leq G$: maximal among the solvable subgroups of G, with respect to inclusion.
  - ($M \leq G$ solvable such that if $M \leq M_0 \leq G$ with M_0 solvable, then $M = M_0$.)
- Examples:
  - Upper triangular matrices in $\mathrm{GL}_n(\mathbb{C})$, for example

$$\left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{C} \right\}$$

    is maximal solvable in $\mathrm{GL}_2(\mathbb{C})$.
  - (More generally, Borel subgroups)
  - Point stabilizer S_4 is maximal solvable in S_5.
  - affine group $\mathrm{AGL}_1(\mathbb{F}_5)$ is maximal solvable in S_5.
- For some classes of groups the following property holds:
  - If $H \leq G$ is solvable, then $H \leq M$ for some maximal solvable subgroup $M \leq G$.
  - This is true for finite groups (obvious), also for linear groups (Zassenhaus 1937)

Introduction

- **Origin:** "Abel's problem": When is $f(X) = 0$ solvable by radicals? (Here e.g. $f \in \mathbb{Q}[X]$.)

"...je me suis aussi occupé de la solution du problème général suivant: Trouver toutes les équations qui sont résoluble algébriquement. Je ne l'ai pas encore achevée, mais autant que j'en puis juger, j'y réussirai." (Abel, letter to B. M. Holmboe, 16 January 1826)

- **Abel:** If $\mathrm{Gal}(f)$ is commutative, then $f(X) = 0$ is solvable by radicals.
- **Galois:** $f(X) = 0$ is solvable by radicals if and only if $\mathrm{Gal}(f)$ is solvable.

Introduction

- **Jordan:** Thesis 1860, and after work during 1860–1870: *Traité des substitutions et des équations algébriques* (1870).
- "l'objet principal" of the Traité:
  - classification of "the general types of equations solvable by radicals"
  - Jordan interprets this as equivalent to the following.
  - **Problem:** Let $N > 0$. Classify the transitive maximal solvable subgroups of S_N (up to conjugacy).
- Assume $f \in \mathbb{Q}[X]$ irreducible. Then $\mathrm{Gal}(f) \leq S_N$ transitive with $N = \deg(f)$.
- Clearly $\mathrm{Gal}(f)$ is solvable if and only if $\mathrm{Gal}(f)$ is contained in some transitive maximal solvable subgroup $M \leq S_N$.
- By resolvent polynomial calculations, we can check whether $\mathrm{Gal}(f)$ is contained in a conjugate of $M \leq S_N$ (check if a certain resolvent polynomial $R_M^f(Y) \in \mathbb{Q}[Y]$ has a rational root).

Overview of results

- **Jordan** (1860-1870, 1908, 1917): classification of maximal solvable subgroups of S_N.
  - Reduces to the classification of maximal irreducible solvable subgroups (MIRS) of $\mathrm{GL}_d(p)$, for p prime.
  - MIRS subgroups of $\mathrm{GSp}_n(r)$, $\mathrm{Sp}_n(r)$, and $O_n^{\pm}(2)$ (r prime, n even).
- **D. A. Suprunenko** (1950s, 1960s): results on maximal solvable subgroups of $\mathrm{GL}_n(\mathbb{F})$, where $\mathbb{F}$ is a field.
  - Description of general structure of MIRS subgroups of $\mathrm{GL}_n(\mathbb{F})$.
- Various other results, e.g. for classical groups in prime degree. (Detinko), computational results (Short 1992, Eick-Höfling 2003)
- (K. 2024) complete classification of maximal irreducible solvable subgroups in $\mathrm{GL}_n(q)$, $\mathrm{GSp}_n(q)$, $\mathrm{Sp}_n(q)$, $\mathrm{GO}_n^{\pm}(q)$, $O_n^{\pm}(q)$.
- (K. 2024) Generators for maximal irreducible solvable subgroups of $\mathrm{GL}_n(q)$ e.g. for $n \leq 127$ in Magma (MIRS on GitHub)
- Details: Springer LNM, Volume 2346. (viii+296 pp.)

Recursive construction

Gives a list which contains all maximal solvable subgroups of S_N up to conjugacy, and **most** subgroups on the list are maximal solvable.

intransitive solvable in S_N

$\rightarrow$ transitive solvable in S_N

$\rightarrow$ primitive transitive solvable in S_N, $N = p^d$ prime power

$\rightarrow$ irreducible solvable in $\mathrm{GL}_d(p)$

$\rightarrow$ primitive solvable in $\mathrm{GL}_d(p)$

$\rightarrow$ irreducible solvable in $\mathrm{GSp}_{2\ell}(r)$ (r prime), and in $O^{\pm}_{2\ell}(2)$

$\rightarrow$ primitive irreducible solvable in $\mathrm{GSp}_{2\ell}(r)$ (r prime), and in $O^{\pm}_{2\ell}(2)$

$\rightarrow$ ...

Bottom of recursion: Singer cycle normalizers $\Gamma\mathrm{L}_1(p^d)$ in $\mathrm{GL}_d(p)$, and their analogues in $\mathrm{GSp}_{2\ell}(r)$, $O^{\pm}_{2\ell}(2)$.

Maximal solvable subgroups of $S_N = \mathrm{Sym}(\Omega)$: intransitive

- Let $M \leq S_N = \mathrm{Sym}(\Omega)$ be maximal solvable. Suppose that M is intransitive.
- Then $\Omega = \Omega_1 \cup \cdots \cup \Omega_t$ disjoint union of M-orbits, where $t \geq 2$.
- We have $M \leq M_1 \times \cdots \times M_t$, where M_i is the action of M on Ω_i.
- Thus $M = M_1 \times \cdots \times M_t$, since M is maximal solvable.
- Furthermore, by maximality M_i is maximal transitive solvable in $\mathrm{Sym}(\Omega_i)$ for all $1 \leq i \leq t$.
- For $i \neq j$, we cannot have $M_i \cong M_j$ as permutation groups, since $M_i \times M_i \lneqq M_i \wr S_2$.
- For similar reasons $M_i \ncong M_j \wr S_2$ and $M_i \ncong M_j \wr S_3$:

$$M_i \times (M_i \wr S_2) < M_i \wr S_3$$
$$M_i \times (M_i \wr S_3) < M_i \wr S_4$$

Maximal solvable subgroups of $S_N = \mathrm{Sym}(\Omega)$: intransitive

- Let $M \leq S_N$ be maximal solvable. If M is intransitive, the following hold:
  (i) $N = n_1 + \cdots + n_t$, where $n_i > 0$ are the sizes of M-orbits on Ω.
  (ii) $M = M_1 \times \cdots \times M_t$, where $M_i \leq S_{n_i}$ is maximal transitive solvable for all i.
  (iii) $M_i \not\cong M_j$ as permutation groups for all $i \neq j$.
  (iv) $M_i \not\cong M_j \wr S_2$ as permutation groups for all $i \neq j$.
  (v) $M_i \not\cong M_j \wr S_3$ as permutation groups for all $i \neq j$.
- **Jordan:** conversely if (i) – (v) hold, then M is a maximal solvable subgroup of S_N.
- Thus the classification of maximal solvable subgroups reduces to the transitive case.

Maximal solvable subgroups of $S_N = \mathrm{Sym}(\Omega)$: imprimitive

- Let $M \leq S_N = \mathrm{Sym}(\Omega)$ be maximal solvable. Suppose that M is transitive, and imprimitive.
- Then $\Omega = \Omega_1 \cup \cdots \cup \Omega_k$ (disjoint union), such that M acts transitively on $\{\Omega_1, \ldots, \Omega_k\}$, and $k > 1$.
- We have $M \leq A \wr B$, where:
  - $A \leq \mathrm{Sym}(\Omega_1)$ is the action of $N_M(\Omega_1)$ on Ω_1.
  - $B \leq \mathrm{Sym}(k)$ is the action of M on $\{\Omega_1, \ldots, \Omega_k\}$.
  - Since M is transitive, both A and B are transitive.
- Because M is maximal solvable, we have $M = A \wr B$.
- Furthermore, by maximality A and B are maximal transitive solvable.
- Repeating this argument on A and B and proceeding inductively, we find that

$$M = M_1 \wr \cdots \wr M_t,$$

with M_i primitive maximal solvable.

Maximal solvable subgroups of $S_N = \mathrm{Sym}(\Omega)$: imprimitive

- Suppose that $M \leq S_N$ is imprimitive transitive maximal solvable. Then the following hold:
  (i) $M = M_1 \wr M_2 \wr \cdots \wr M_t$.
  (ii) $N = n_1 n_2 \cdots n_t$.
  (iii) $M_i \leq S_{n_i}$ primitive maximal solvable, $n_i > 1$ prime power, for all $1 \leq i \leq t$.
  (iv) $(n_i, n_{i+1}) \neq (2,2)$ for all $1 \leq i < t$.
- Condition (iv) is needed, since $S_2 \wr S_2 \lneq S_4$.
- **Jordan:** conversely if (i) – (iv) hold, then M is a maximal solvable subgroup of S_N.
- Example: $S_2 \wr S_4 \wr S_2$ is a maximal solvable subgroup of S_{16}.
- Example: $S_2 \wr S_2 \wr S_4$ is not a maximal solvable subgroup of S_{16}, since

$$(S_2 \wr S_2) \wr S_4 < S_4 \wr S_4 < S_{16}.$$

Maximal solvable subgroups of $S_N = \mathrm{Sym}(\Omega)$: primitive

- Suppose that $M \leq S_N$ is maximal solvable. Suppose that M is transitive and primitive.
- Then $N = p^d$ for some prime p, and M is an affine group

$$(\mathbb{F}_p)^d \rtimes X,$$

  where $X \leq \mathrm{GL}_d(p)$ is maximal irreducible solvable.
- **Jordan:** Conversely, such an affine group is always maximal solvable in S_N.
- Thus the classification of maximal solvable subgroups of S_N reduces to the classification of maximal irreducible solvable subgroups of $\mathrm{GL}_d(p)$, for p prime.

MIRS subgroups of $\mathrm{GL}_n(q)$: imprimitive case

- An irreducible subgroup $M \leq \mathrm{GL}_n(q)$ is imprimitive, if there is a decomposition
$$(\mathbb{F}_q)^n = W_1 \oplus \cdots \oplus W_k$$
such that $k > 1$, and M acts transitively on $\{W_1, \ldots, W_k\}$.
- Imprimitive MIRS is of the form $M = H \wr K = (H \times \cdots \times H) \rtimes K$:
  - $n = dk$
  - $H \leq \mathrm{GL}_d(q)$ primitive MIRS
  - $K \leq S_k$ maximal transitive solvable
  - $q > 2$ or $d > 1$
- Conversely, every such $M = H \wr K$ is irreducible and solvable (but not always maximal solvable).
- **Example:** $\mathrm{GL}_1(q) \wr S_2$ is MIRS in $\mathrm{GL}_2(q)$ if and only if $q = 4$ or $q > 5$.
- **Example:**
$$\mathrm{GL}_1(3) \wr S_4 < 2_+^{1+4}.O_4^+(2) < \mathrm{GL}_4(3),$$
so $\mathrm{GL}_1(3) \wr S_4$ is not MIRS in $\mathrm{GL}_4(3)$.

MIRS subgroups of $\mathrm{GL}_n(q)$: primitive case

- Let $M \leq \mathrm{GL}_n(q)$ be primitive maximal irreducible solvable.
- M is solvable, so there is a normal series

$$\{1\} = M_0 \trianglelefteq M_1 \trianglelefteq M_2 \trianglelefteq \cdots \trianglelefteq M_t = M$$

  with M_i/M_{i+1} abelian for all $1 \leq i < t$, and $M_i \trianglelefteq M$.
- Bottom-up construction: first describe M_1, then M_2, then ...

MIRS subgroups of $\mathrm{GL}_n(q)$: primitive case

- Let $M \leq \mathrm{GL}_n(q)$ be primitive maximal irreducible solvable.
- **Outline:**
  1. Describe the possible structures of a maximal abelian normal subgroup $F \trianglelefteq M$.
  2. Describe the structure of $N_{\mathrm{GL}_n(q)}(F)$.
  3. Describe the structure of $F \trianglelefteq A \trianglelefteq M$ such that:
     - ★ A/F is abelian;
     - ★ $A \leq C_M(F)$;
     - ★ A is maximal with respect to these two properties.
  4. Describe the structure of $N_{\mathrm{GL}_n(q)}(F, A)$.
  5. ...
- (Turns out both F and A are uniquely determined.)

MIRS subgroups of $\mathrm{GL}_n(q)$: primitive case

- Let $M \leq \mathrm{GL}_n(q)$ be primitive maximal irreducible solvable.
- Consider a maximal abelian normal subgroup $F \trianglelefteq M$.
- Then F is the multiplicative group of a finite field $\mathbb{F}_{q^d}$, where $n = d\mu$ for some integer $\mu \geq 1$.
  - ($\mathbb{F}_{q^d} = F \cup \{0\}$ is an $\mathbb{F}_q$-subalgebra of $\mathrm{Mat}_n(q)$.)
- If $\mu = 1$, then $M = \Gamma\mathrm{L}_1(q^n)$ (normalizer of a Singer cycle).
- Suppose $\mu > 1$, with prime factorization $\mu = r_1^{\ell_1} \cdots r_k^{\ell_k}$.
- Then $r_i \mid q^d - 1$ for all $1 \leq i \leq k$. Define $A = \mathrm{Fit}(C_M(F))$.
- $C_M(F) \leq \mathrm{GL}_\mu(q^d)$, and A is an absolutely irreducible subgroup of $\mathrm{GL}_\mu(q^d)$.
- $A = R_1 \otimes \cdots \otimes R_k$, with $R_i \leq \mathrm{GL}_{r_i^{\ell_i}}(q^d)$ absolutely irreducible.
- Here $R_i \trianglelefteq M$ is extraspecial of order $r_i^{1+2\ell_i}$. (And exponent r_i if $r_i > 2$.)

Maximal irreducible solvable subgroups of $\mathrm{GL}_n(q)$: primitive case

- $F \trianglelefteq A \trianglelefteq M$
- $F = \mathbb{F}_{q^d}^{\times}$, $n = d\mu$
- $A = R_1 \otimes \cdots \otimes R_k$ with $R_i \trianglelefteq M$
- $R_i/Z(R_i) \cong (\mathbb{F}_{r_i})^{2\ell_i}$ (symplectic space)

- We have a map

$$\pi : N_{\mathrm{GL}_n(q)}(F, R_1, \ldots, R_k) \to \mathrm{GSp}_{2\ell_1}(r_1) \times \cdots \times \mathrm{GSp}_{2\ell_k}(r_k)$$

with $\pi(g) = (g_1, \ldots, g_k)$, where g_i is the action of g on $R_i/Z(R_i)$.

- $\mathrm{Ker}\,\pi$ is solvable, so $M = \pi^{-1}(X_1 \times \cdots \times X_k)$, where $X_i \leq \mathrm{GSp}_{2\ell_i}(r_i)$ is maximal solvable.

Maximal irreducible solvable subgroups of $\mathrm{GL}_n(q)$: primitive case

- If $G \leq \mathrm{GL}_n(q)$ is primitive maximal irreducible, then $G = \pi^{-1}(X_1 \times \cdots \times X_k)$, where $X_i \leq \mathrm{GSp}_{2\ell_i}(r_i)$ is maximal solvable. Denote
$$G = G^{\mathcal{B}_0}_{\mu,\nu}(X_1, \ldots, X_k).$$
- Furthermore, each X_i is *metrically completely reducible*, meaning that it has no non-zero totally isotropic invariant subspaces.
- If $X \leq \mathrm{GSp}_{2\ell}(r)$ is metrically completely reducible maximal solvable, then
$$X = (Y_1 \times \cdots \times Y_s) \cap \mathrm{GSp}_{2\ell}(r)$$
where $Y_i \leq \mathrm{GSp}_{2k_i}(r)$ is maximal irreducible solvable for all i, and $k_1 + \cdots + k_s = \ell$.
- Similar analysis for GSp, Sp, GO, ...

References

This talk:

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See also:

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